

# Outer space and Automorphisms of free groups

## LECTURE 4

Last time we defined Outer space  $CV_n$  in three different ways- as a space of

1. actions on trees
2. Marked graphs
3. Sphere systems in  $M_n = \#_n S^1 \times S^2$

Today will prove  $CV_n$  is contractible.

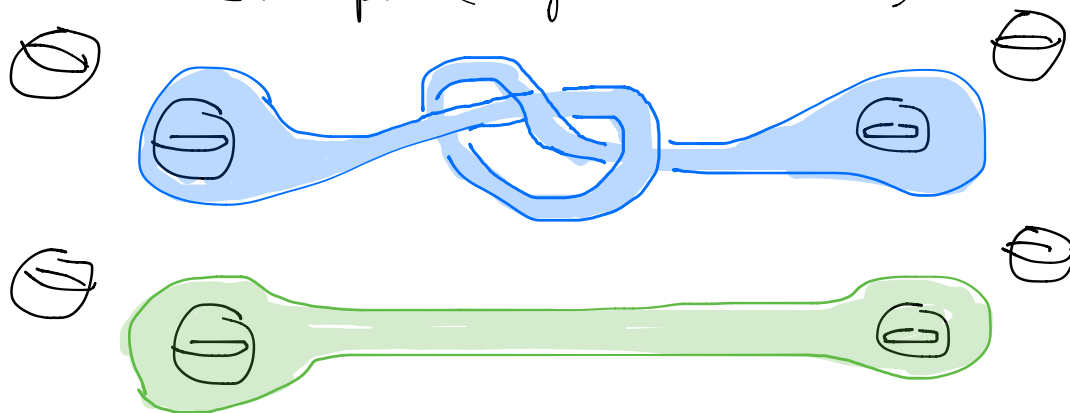
There are proofs using all three models.

All proofs have a common idea: fix a point  $X_0 \in CV_n$  then retract all of  $CV_n$  to  $X_0$  by following paths which reduce some measure of complexity.

One of the most natural - due to Hatcher -  
uses the sphere system model

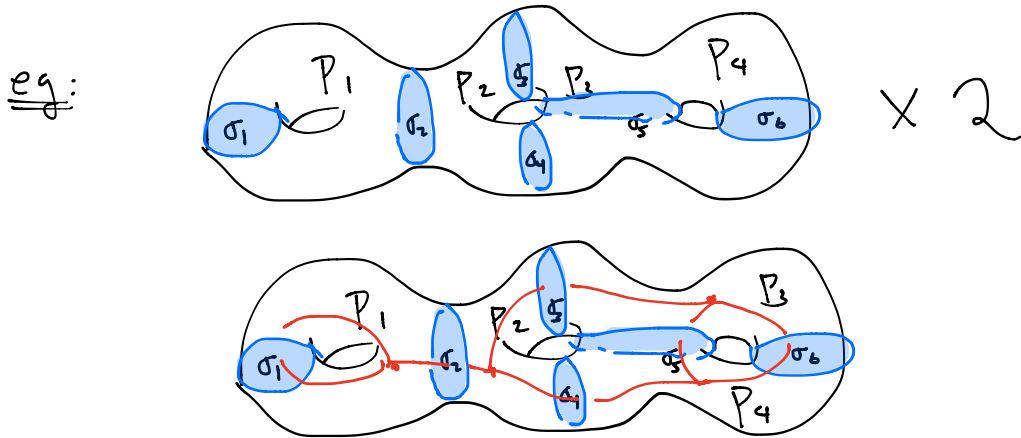
This requires some non-trivial results from  
3-manifold theory, based ultimately on  
Laudenbach's theorem that in  $M_n = \#_n S^1 \times S^2$   
homotopic sets of embedded 2-spheres are isotopic

Eg.: the green and blue spheres below  
are isotopic ("light bulb trick")

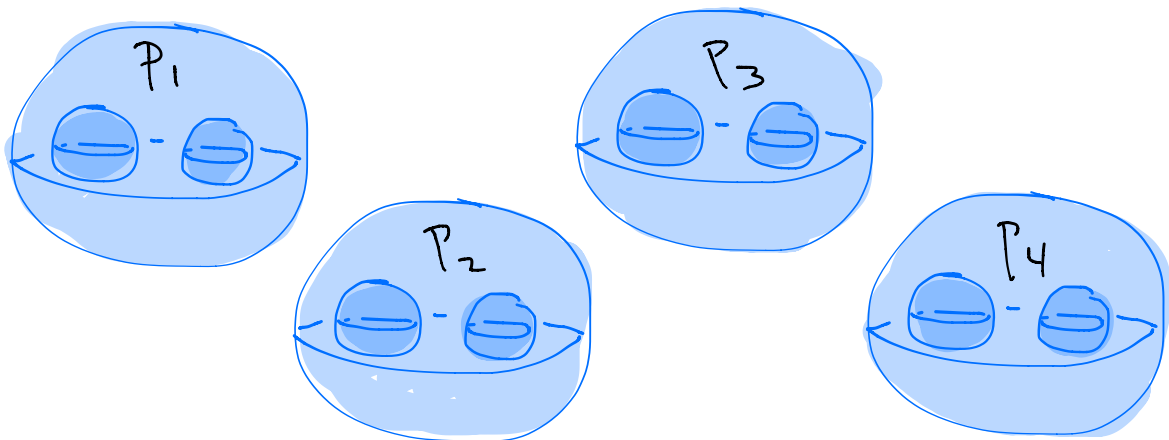


It's not so obvious when an entire sphere  
system is knotted and linked!

Fix  $\Sigma = \{\sigma_1, \dots, \sigma_{3n-3}\}$  a maximal sphere system in  $M_n$ .  
 $(\leftrightarrow$  trivalent marked graph  $(g, G)$ )



$\Sigma$  cuts  $M_n$  into 3-punctured spheres:  $P_i = S^2 \setminus (B_1 \cup B_2 \cup B_3)$



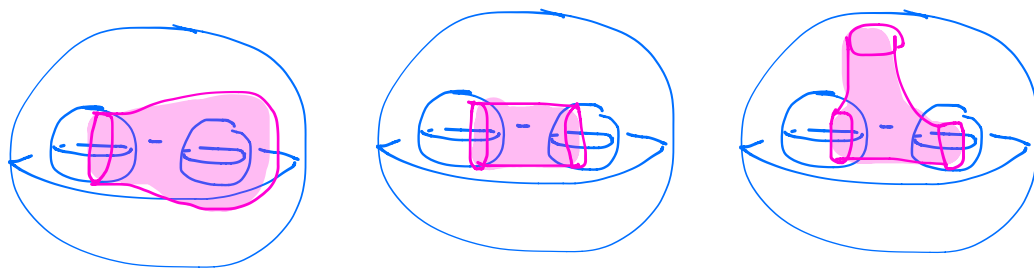
So  $\Sigma$  is simple, gives simplex  $|\Sigma|$  in  $CV_n$   
 (Recall open simplex in  $CV_n \leftrightarrow$  simple sphere system  
 face  $\leftrightarrow$  simple sub-system)

We will retract all of  $S(M)$  to this simplex  $|\Sigma|$

Let  $\mathcal{S}$  be another sphere system  $= \{s_1, \dots, s_k\}$

$\mathcal{S}$  is in normal form wrt  $\Sigma$  if

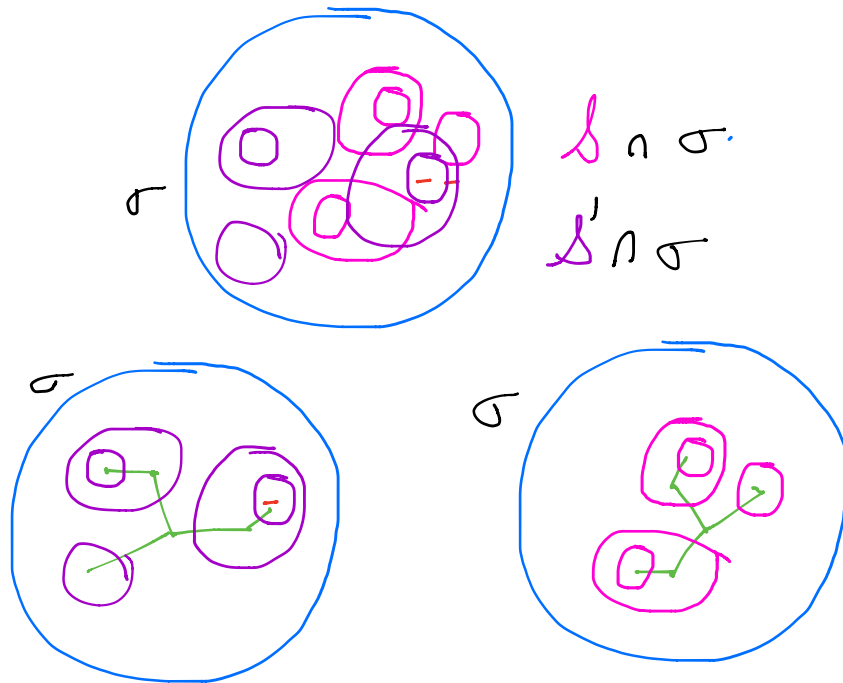
- Each  $s \in \mathcal{S}$  intersects  $\Sigma$  transversely in a finite number of circles, which cut  $s$  into **pieces**
- Each piece is a disc, cylinder or pair of pants, w/ at most one  $\partial$  circle on each component of  $\partial P_i$ :



### Hatcher's normal form theorem

1. Every sphere system is isotopic to a sphere system in normal form with respect to  $\Sigma$ .
2. If  $\mathcal{S}$  and  $\mathcal{S}'$  are both in normal form and are isotopic, then they are isotopic by an

isotopy which preserves the pattern of intersection circles on each  $\sigma \in \Sigma$ .



(The pattern is encoded in the dual tree)

The proof relies heavily on Landenbach's theorem

Prop  $\mathcal{L}$  is in normal form wrt  $\Sigma$  if and only if  $\mathcal{L}$  is transverse to  $\Sigma$  and the number of intersection circles  $\mathcal{L} \cap \Sigma$  is minimal.

(If it's not in normal form, you can reduce the # of intersection circles.)

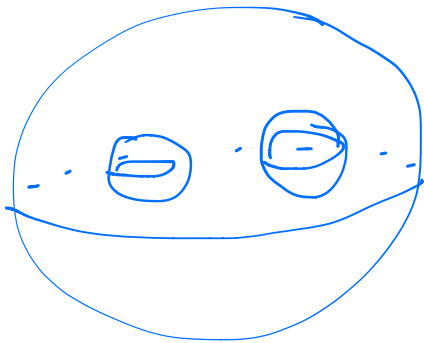
A point in  $S(M)$  is given by barycentric coords = weights on the spheres  $s_i \in \mathcal{L}$

We want a path from a weighted sphere system  $\mathcal{L}$  to a point of  $|\Sigma|$ .

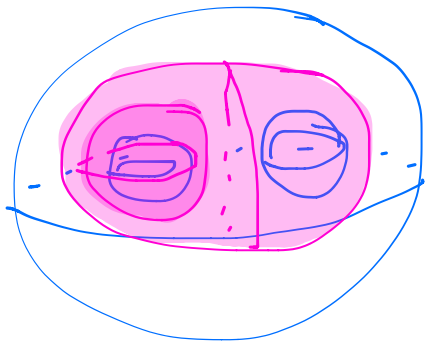
Idea: Do (weighted) surgery on  $\mathcal{L}$  using innermost disks on spheres in  $\Sigma$  to eliminate intersection

circles. First describe unweighted surgery:

If  $\mathcal{L}'$  is disjoint from  $\Sigma$  then  $\mathcal{L}'$  is a subset of  $\Sigma$



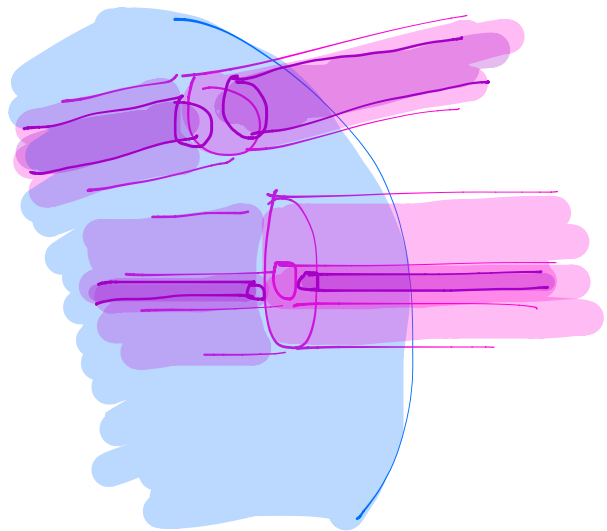
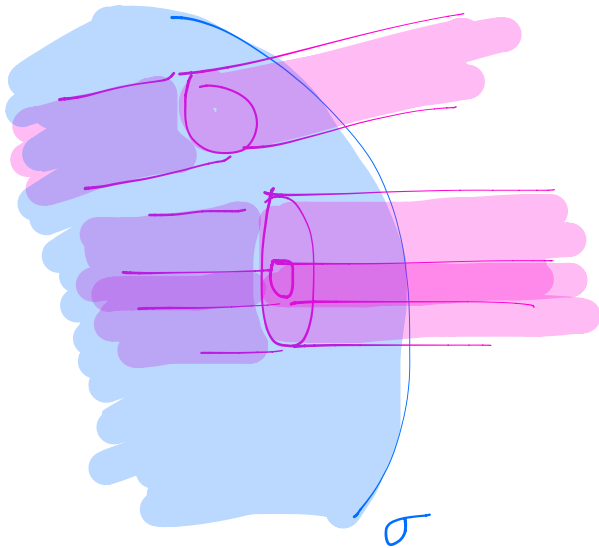
← any sphere in here



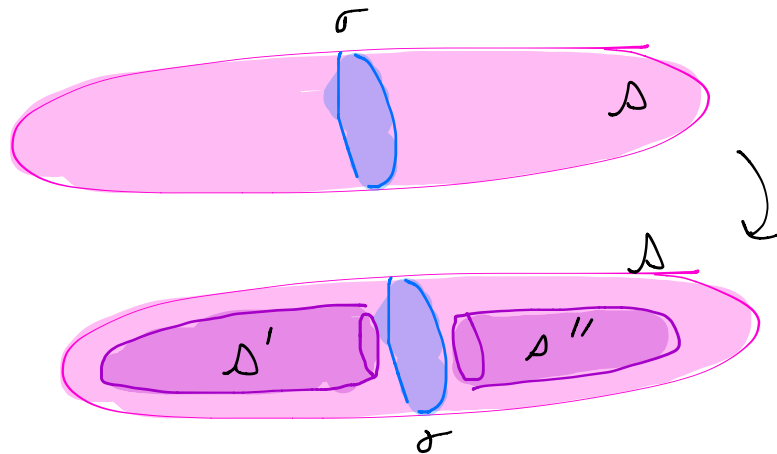
is parallel to one of  
the boundary spheres:



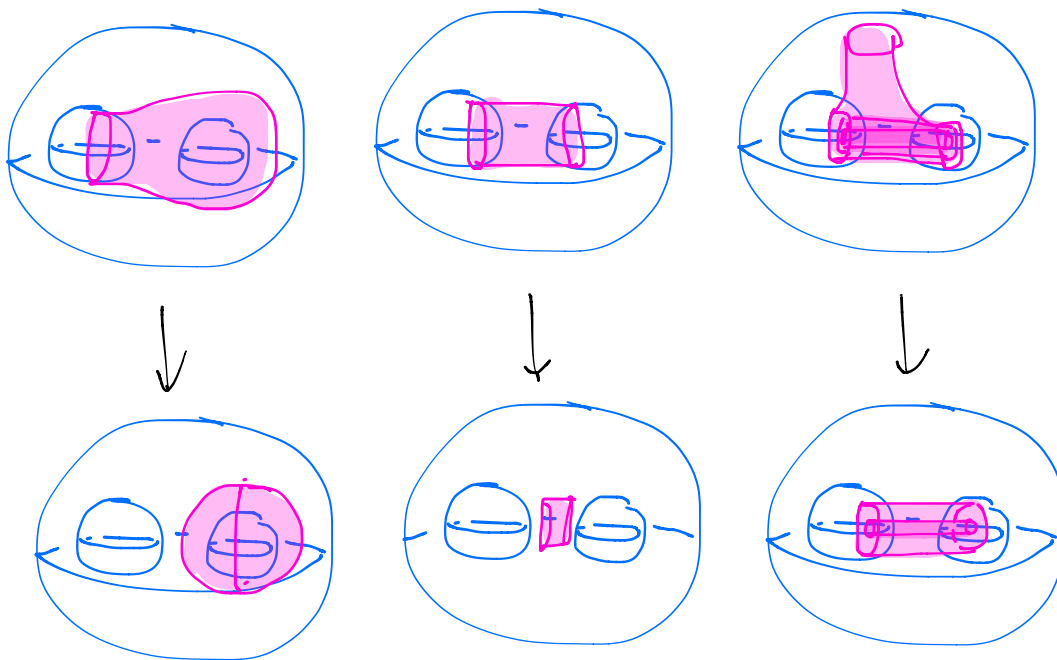
If  $L$  is not disjoint from  $\Sigma$ , do surgery



Another view of a single surgery:



What this does to the pieces of  $S$ :





Each disk piece becomes  $\partial$ -parallel

Each cylinder piece becomes trivial

A pants piece can become a cylinder or trivial.

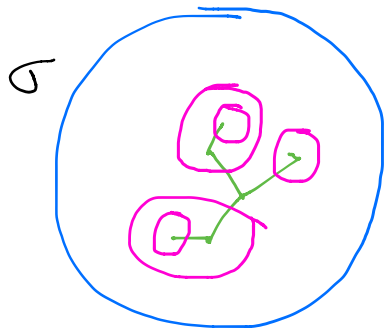
Discard trivial and boundary spheres. The resulting system is still in normal form!

To describe a path from  $\mathcal{S}$  to  $\mathcal{S}'$ : transfer the weight of  $\mathcal{S}$  to  $\mathcal{S}'$

To get a path from  $\mathcal{S}$  to  $|\Sigma|$ : do surgery on all innermost disks simultaneously, transfer weight to new spheres gradually, discard spheres when they have weight zero

If there's only one circle on  $\sigma$ , use both innermost disks (weight will  $\rightarrow 0$  twice as fast).

Can see what is happening by looking at the dual trees in the  $\sigma_i$ :



length of edge = weight  
of dual circle.

Innermost disks  $\leftrightarrow$  leaves

As circle disappears, leaves grow shorter.

then disappear. Process recommences  
with remaining (or new) leaves.

This is a nice intuitive picture, but there is  
work to be done: need to show deformation  
along path  $\mathcal{L} \rightarrow |\Sigma|$  is continuous.

Hatcher uses the dual trees to show that process  
gives a piecewise linear flow on  $S(M_n)$

## Contraction of $CV_n$ :

We have  $S(M_n) =$  sphere complex  
 $=$  simplicial closure  $CV_n^*$

Simplex in  $S(M_n) =$  sphere system

point in  $S(M_n) =$  sphere system w/  
barycentric coords

$=$  weighted sphere system

$CV_n \subset S(M_n) =$  weighted simple sphere  
systems

Exercise: If  $\mathcal{S}$  is simple, then so is

$$\mathcal{S}' = \mathcal{S} - \{s\} \cup \{s', s''\} \quad (\text{Van Kampen's Theorem})$$

So Hatcher's retraction restricts to a retraction of  
 $CV_n$  to a point.

Next: The spine of  $CV_n$

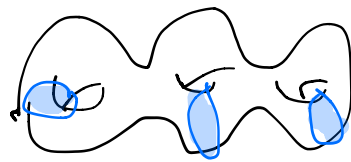
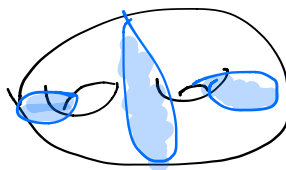
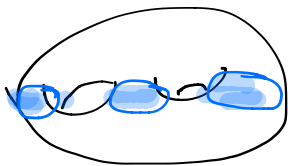
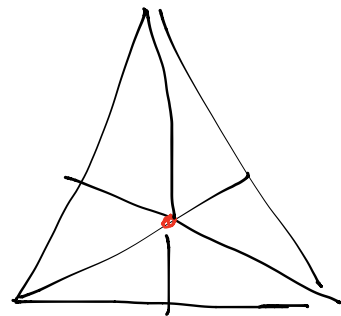
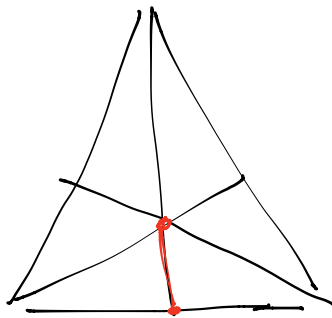
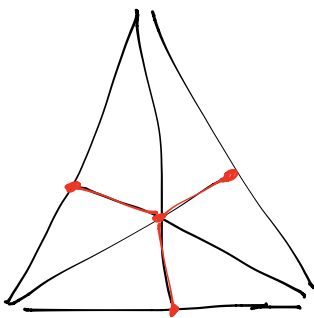
$S'(M_n) =$  barycentric subdivision of  $S(M_n)$

vertices = sphere systems

$k$ -simplex = chain  $\delta_0 \subset \delta_1 \subset \dots \subset \delta_k$

Definition  $K_n \subset S'(M_n)$  is the subcomplex spanned by simple sphere systems.

= Spine of Outer space



$\text{Out}(F_n)$  acts on  $K_n$ , since diffeomorphisms preserve the property of being simple

Furthermore,  $CV_n$  retracts onto  $K_n$ :

$\Delta = \{\Delta_0, \dots, \Delta_k\}$  a simple sphere system

$x = a_0 \Delta_0 + \dots + a_k \Delta_k = \text{point in } CV_n$

is in  $\{\Delta\} \subset \{\Delta_0, \Delta_1\} \subset \dots \subset \{\Delta_0, \dots, \Delta_k\} = \Delta$  some simplex of  $S'(M_n)$

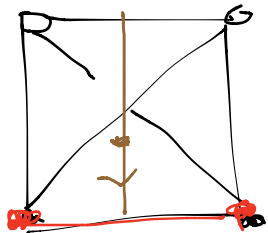
$\Delta_0 \subset \Delta_1 \subset \dots \subset \Delta_i \subset \dots \subset \Delta_k = \Delta$

$x = b_0 \Delta_0 + b_1 \Delta_1 + \dots + b_k \Delta_k$

If  $\Delta_i$  is the first simple system, move  $x$  to the point

$b_i \Delta_i + \dots + b_k \Delta_k$  (normalized so  $\sum b_j = 1$ )

by uniformly shrinking  $b_1, \dots, b_{i-1}$  and expanding  $b_i, \dots, b_k$



Exercise: The retractions on an open simplex in  $CV_n$  extends continuously to a face which is in  $CV_n$ .

Corollary:  $K_n$  is contractible

Proposition:  $\dim(K_n) = 2n - 3$

pf: A simple sphere system needs at least  $n$  spheres and can have at most  $3n - 3$

So the longest chain has length  $2n - 2$

So the largest simplex in  $K_n$  has dimension  $2n - 3$ .

The stabilizer of a (weighted) sphere system  $\mathcal{S}$

permutes the spheres. Since an orientation-preserving diffeomorphism of a punctured sphere which fixes the boundary is

isotopic to the identity, the stabilizer

is finite.

The quotient is compact:

Up to diffeomorphism there are only finitely many sphere systems in  $M_n$  (there are only finitely many ways to glue together punctured balls to obtain  $M_n$ , by  $X$ ).

We now have a contractible,  $(2n-3)$ -dimensional simplicial complex  $K_n$  on which  $\text{Out}(F_n)$  acts with finite stabilizers.

Algebraic Consequences:

Thm (Hurewicz)  $X = \text{CW-complex}$ ,  $G = \pi_1 X$ .

If  $\tilde{X}$  is contractible, then  $H^*(X)$  is an invariant of  $G$ .

This is one way to define the cohomology of a group.  $X$  is called a  $K(G, 1)$ .

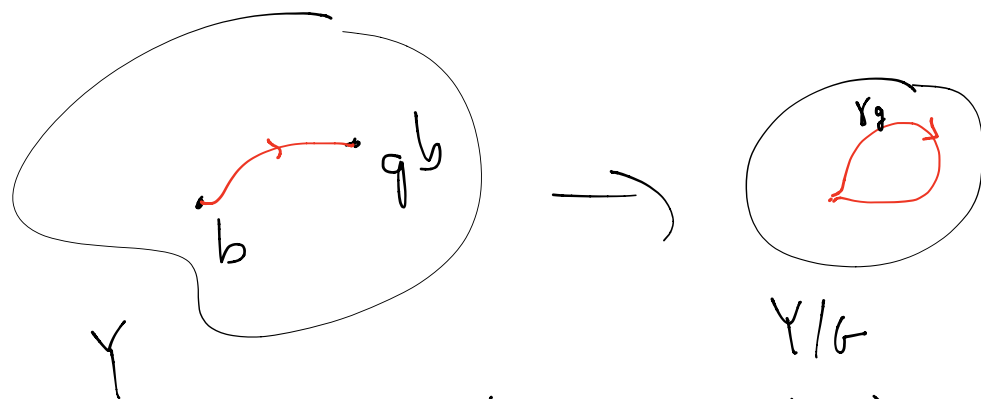
There are also purely algebraic definitions - see Ken Brown's excellent book, *Cohomology of Groups*.

How to find such an  $X$ ?

Note  $\pi_1 X$  acts freely on  $\tilde{X}$

Conversely, if  $G$  acts freely on a contractible  $Y$ ,

then the map  $G \rightarrow \pi_1(Y/G)$



is an isomorphism. (Algebraic topology)



We have  $K_n$  contractible,  $\text{Out}(F_n)$  acting,  
but the action is not free, stabilizers are finite

Thm: (Baumslag-Taylor)  $\text{Out}(F_n)$  has finite-index  
subgroups with no torsion

Let  $\Gamma < \text{Out}(F_n)$  be such a subgroup

$\Gamma$  acts on  $K_n$ , too, and contains no stabilizers,  
so  $\Gamma$  acts freely!

$$\text{so } H^*(K_n/\Gamma) = H^*(\Gamma)$$

We now get:

Thm  $H^i(\Gamma) = 0$  for  $i > 2n-3$  ( $= \dim K_n$ )

Turns out the choice of  $\Gamma$  is irrelevant for this:

Thm (Serre)  $\Gamma, \Gamma'$  torsion-free finite index subgroups of  $G \Rightarrow$  cohomological dimension of  $\Gamma =$  cohomological dimension of  $\Gamma'$ .

Called the virtual cohomological dimension of  $G$ .

Also,  $K_n/\text{out}(F_n)$  is a finite CW complex (finitely many cells) so  $K_n/\Gamma$  has the same property ( $\Gamma$  is finite index) and we get

Thm  $H^i(\Gamma)$  is finitely generated for all  $i$ .